EE 632: Advanced Topics in Communications

Homework 3

Question 1. Consider a discrete finite memory stationary causal channel with $s[i]$, for $i = 1, 2, \ldots, t$, as the transmit symbols drawn from a constellation of size m with equal probability. Let $y[i]$ denote the received symbols at the channel output given by

$$y[i] = \sqrt{\rho} \sum_{\tau=1}^{3} e^{-\gamma(\tau-1)} s[i-\tau+1] + w[i], \text{for } i = 1, 2, \ldots, t, \tag{1}$$

where γ denote the channel constant and $w[i]$ denote the i.i.d. noise with distribution $\mathcal{CN}(0, \sigma^2)$ and transmit symbols with negative time indices are zeros.

1. Find the distribution of $y[1]$ conditioned on $s[1]$.
2. Find the distribution of $y[3]$ conditioned on $s[1], s[2], s[3]$.
3. Find the joint probability $\Pr\left(y[1], y[2], \ldots, y[t] | s[1], s[2], \ldots, s[t]\right)$ and compute

$$\log\left(\Pr\left(y[1], y[2], \ldots, y[t] | s[1], s[2], \ldots, s[t]\right)\right).$$

Question 2. Let $\mathbf{x}$ denote the vector of parameters to be estimated and $\mathbf{y}$ denote the vector of observations. They are jointly Gaussian with joint distribution as below

$$\begin{bmatrix} \mathbf{y} \\ \mathbf{x} \end{bmatrix} \sim \mathcal{N}(\boldsymbol{\mu}, \mathbf{C}),$$

where

$$\boldsymbol{\mu} = \begin{bmatrix} \boldsymbol{\mu}_y \\ \boldsymbol{\mu}_x \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} \mathbf{C}_{yy} & \mathbf{C}_{yx} \\ \mathbf{C}_{xy} & \mathbf{C}_{xx} \end{bmatrix}.$$

Show that the MMSE estimator of $\mathbf{x}$ given observation $\mathbf{y}$ and covariance matrix of the error vector are given by

$$\hat{\mathbf{x}}_{\text{MMSE}} = \mu_x + \mathbf{C}_{xy}\mathbf{C}_{yy}^{-1}\left(y - \mu_y\right), \tag{2}$$

$$\mathbf{C}_{\text{Err}} = \mathbf{C}_x - \mathbf{C}_{xy}\mathbf{C}_{yy}^{-1}\mathbf{C}_{yx}. \tag{3}$$

Question 3. Consider a cell-free massive MIMO system with L APs each equipped with N antennas serving $K = \tau_p + 1$ UEs. Let $\phi_1, \ldots, \phi_{\tau_p} \in \mathbb{C}^{\tau_p \times 1}$ denote the unit norm orthogonal pilot vectors, i.e., $\|\phi_k\| = 1$, for $k = 1, \ldots, \tau_p$ and $\phi_i^H \phi_k = 0$, for $i \neq k$. Let $\mathbf{h}_{il} \in \mathbb{C}^{N \times 1}$ denote the channel gain vector from the i^{th} UE to the l^{th} AP, where $i = 1, \ldots, K$ and $l = 1, \ldots, L$.

The UEs 1 and 2 transmit the same pilot vector ϕ_1 and the UEs $3, \ldots, K$ transmit the remaining orthogonal pilot vectors each. The received pilot matrix at the l^{th} AP is given by

$$\mathbf{Y}_l = \sqrt{\eta_1}\mathbf{h}_{1l}\phi_1^H + \sqrt{\eta_2}\mathbf{h}_{2l}\phi_1^H + \sum_{i=3}^{K} \sqrt{\eta_i}\mathbf{h}_{il}\phi_{i-1}^H + \mathbf{N}_l, \tag{4}$$

where η_i, for $i = 1, \dots, K$, denote the pilot power allocated by the i^{th} UE and $\mathbf{N}_l \in \mathbb{C}^{N \times \tau_p}$ has i.i.d. $\mathcal{CN}\left(0, \sigma_{ul}^2\right)$ entries that denote additive white Gaussian noise.

1. Show that the distribution of $\mathbf{n}_i = \mathbf{N}_l \phi_i \sim \mathcal{CN}\left(\mathbf{0}, \sigma_{ul}^2 I_N\right)$.

2. For uncorrelated Rayleigh fading, i.e., $\mathbf{h}_{kl} \sim \mathcal{CN}\left(\mathbf{0}, \beta_{kl} I_N\right)$ and under the assumption that channel gains from different users to the AP l are independent, Show that the MMSE estimates

$$\hat{\mathbf{h}}_{1l} = \frac{\sqrt{\eta_1}\beta_{1l}}{\eta_1\beta_{1l} + \eta_2\beta_{2l} + \sigma_{ul}^2}\mathbf{Y}_l\phi_1, \tag{5}$$

$$\hat{\mathbf{h}}_{3l} = \frac{\sqrt{\eta_3}\beta_{3l}}{\eta_3\beta_{3l} + \sigma_{ul}^2}\mathbf{Y}_l\phi_3. \tag{6}$$