EE-668: Massive MIMO for 5G Communications: Design and Analysis

Homework 1

Question 1. Given a n-dimensional standard Gaussian random vector $\mathbf{w} \sim \mathcal{N}(\mathbf{0}, \mathbf{I}_n)$, prove that its squared norm $||\mathbf{w}||^2$ is chi-square distributed with n degrees of freedom. For an orthogonal matrix $\mathbf{Q}$ (i.e., $\mathbf{Q}^T\mathbf{Q} = \mathbf{Q}\mathbf{Q}^T = \mathbf{I}$), show that $\mathbf{Qw}$ is also standard Gaussian random vector.

Question 2. Given a Gaussian random vector $\mathbf{x} \sim \mathcal{N}(\mu, \mathbf{C})$, show that the components of the random vector $\mathbf{y} = \mathbf{Ax}$ (assume $\mathbf{A}^{-1}$ exists) are jointly Gaussian. Use the transformation of RVs to find the joint pdf of the components in $\mathbf{y}$ and show that it takes a Gaussian form.

Question 3. Let $w(t)$ be white Gaussian noise with power spectral density $\frac{N_o}{2}$. Let $\mathbf{s}_1, \ldots, \mathbf{s}_M$ be a set of finite orthonormal waveforms (i.e., orthogonal and unit energy), and define $z_i = \int_{-\infty}^{\infty} w(t)s_i(t)dt$. Find the joint distribution of $\mathbf{z}$.

Question 4. Let $\mathbf{x}$ be an n-dimensional i.i.d complex Gaussian random vector. The real and imaginary parts of its components are distributed as $\mathcal{N}(\mathbf{0}, \mathbf{K}_x)$ where $\mathbf{K}_x$ is a 2×2 covariance matrix. Suppose $\mathbf{U}$ is a unitary matrix (i.e., $\mathbf{U}^H\mathbf{U} = \mathbf{U}\mathbf{U}^H = \mathbf{I}$). Identify the conditions on $\mathbf{K}_x$ under which $\mathbf{Ux}$ has the same distribution as $\mathbf{x}$.

Question 5. If $W = W_r + jW_i$ is circular symmetric complex Gaussian random variable, i.e., $W \sim \mathcal{CN}(0, 1)$, prove that the envelope $r = |W| = \sqrt{W_r^2 + W_i^2}$ is Rayleigh distributed and its power r^2 is exponential distributed.