EE-668: Massive MIMO for 5G Communications: Design and Analysis

Assignment 3

Question 1. Prove below statements

1. Entropy of a random variable is non-negative.
2. Entropy of a random variable X is equal to zero only when it takes one value a with probability 1, i.e., $\Pr(X = a) = 1$.
3. Prove that uniform distribution maximizes the entropy among all the probability distributions $p(x)$ which are nonzero over a finite range $x \in [a, b]$
4. Prove that exponential distribution maximizes the entropy among all the probability distributions $p(x)$ which are nonzero over a semi-infinite range $x \in [0, \infty]$ and has a given mean $\mu < \infty$.

Question 2. Show that the entropy of multivariate Gaussian random vector $\mathbf{x} \sim \mathcal{N}(\mu, \Sigma)$ is given by

$$h[x] = \frac{1}{2}\log_2\left((2\pi e)^n |\Sigma|\right),$$

where n is the dimension of the random vector $\mathbf{x}$.

Question 3. Prove

1. The estimated channel coefficient $\hat{g}_k^m$ and estimation error $\tilde{g}_k^m$ are jointly gaussian.
2. The elements of $\mathbf{N} = \mathbf{W}_p\mathbf{\Phi}$ are i.i.d $\mathcal{CN}(0, 1)$ when $\mathbf{W}_p$ is an $M \times \tau_p$ matrix with i.i.d. $\mathcal{CN}(0, 1)$ elements and $\mathbf{\Phi}$ is a $\tau_p \times K$ matrix such that $\mathbf{\Phi}^H\mathbf{\Phi} = I_K$.

Question 4. For an $m \times n$ matrix $\mathbf{B}$, $n \times m$ matrix $\mathbf{D}$, and $n \times 1$ vectors $\mathbf{a}$ and $\mathbf{b}$, prove the following

1. Optimal $\mathbf{a}$ that maximizes $\frac{|\mathbf{a}^H\mathbf{b}|^2}{\mathbf{a}^H\mathbf{B}\mathbf{a}}$ is given by $\mathbf{B}^{-1}\mathbf{b}$.
2. $B^{-1}a = (1 + a^H B^{-1}a)(B + aa^H)^{-1}a$.
3. $[BD + I_m]^{-1}B = B[DB + I_n]^{-1}$.